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FLOW THROUGH CASCADES IN TANDEM

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SUMMARY

An exact treatment of the problem of finding the incompressible, inviscid two-dimensional flow around two cascades in tandem is presented. The analysis includes solutions of both the direct and the inverse problems. These problems are solved by conformally mapping the tandem cascade onto the region between two concentric circles in which region there are suitably placed flow singularities. Formulas for the velocity and the potential in the annular region are presented in a closed form by means of elliptic functions. The equations are presented in a form suitable for computation.

INTRODUCTION

A theory of the two-dimensional flow of an incompressible inviscid fluid through two cascades in tandem, which was developed at the NACA Lewis laboratory, is presented herein. Methods are developed for the solution of both the direct and the inverse problems for a tandem cascade. For a single cascade, there have been several solutions of these problems. A survey of such problems is presented in reference 1.

The results of this analysis can be applied to a variety of problems that occur in compressors. The amount of turning that can be accomplished by a single row of blades without separation of the blade boundary layer appears to be limited. Criterions for the amount of circulation attainable around a blade of a cascade without separation of the boundary layer, with the attendant increased losses, are given in reference 2. If an amount of turning greater than that which can be obtained with a single blade row without boundary-layer separation is desired, this turning may be accomplished by the use of two rows of blades, which are so designed that blade boundary-layer separation does not take place.

Another situation to which the results of this analysis are applicable is the following: In some compressors, there are additional rows of turning vanes beyond the last row of stator blades. These rows of turning vanes form a configuration to which the present analysis can be applied as a two-dimensional approximation. In particular, indications of flow for angles of attack other than the design angle of attack can be obtained.

A tandem cascade is composed of two cascades arranged approximately as shown in figure 1(a). The method employed herein for the study of this problem is the method of conformal transformation of the tandem cascade into a standard doubly connected region for which the flow is known. Various stages of the transformation are sketched in figure 1. In many respects, the problem of the tandem cascade is similar to that of the biplane. As is apparent in figure 1(b), the first stage of the transformation renders the tandem cascade into what is essentially a two-dimensional biplane. The standard doubly connected region chosen is the same as that used in reference 3 for the biplane problem; that is, the region contained between two concentric circles.

GENERAL EQUATIONS FOR FLOW THROUGH TANDEM CASCADE

In order to determine certain relations between various flow parameters of a tandem cascade as shown in figure 1(a), the following procedure is used. The lines P_1P_4 and P_2P_3 are located far downstream and far upstream of the tandem cascade, respectively, (fig. 1(a)). The streamlines P_1P_2 and P_3P_4 are located exactly one cascade spacing apart; P_1P_4 and P_2P_3 are parallel to the cascade axes.

From the continuity condition, the net flow from the contour $P_1P_2P_3P_4P_1$ (fig. 1(a)) must be zero. Because P_1P_2 and P_3P_4 are streamlines, the flow across them is zero. The flow through P_2P_3 must therefore equal the flow through P_1P_4. From this relation

$$tV_1 \cos \lambda_1 = tV_2 \cos \lambda_2$$

where t is the cascade spacing (the same for both cascades) or

$$V_1 \cos \lambda_1 = V_2 \cos \lambda_2 \tag{1}$$

where V_1 and λ_1 are the limiting values of magnitude and direction of the velocity far upstream of the tandem cascade, respectively; and V_2 and λ_2 are the limiting values of the same quantities downstream of the tandem cascade. (The main symbols used herein are defined in appendix A.)

The circulation around the blades enclosed within the contour is given by

$$\Gamma = \oint_{P_1P_2P_3P_4P_1} u \, ds \tag{2}$$

The integrals along P_1P_2 and P_3P_4, in equation (2), are equal in magnitude and opposite in sign. This integral therefore reduces to

$$\oint_{P_1P_2P_3P_4P_1} u \, ds = t \left(V_1 \sin \lambda_1 - V_2 \sin \lambda_2\right)$$

or

$$\frac{\Gamma}{t} = V_1 \sin \lambda_1 - V_2 \sin \lambda_2 \tag{3}$$

In figure (2), V_m and λ_m refer, respectively, to the magnitude and the direction of the vector mean of the upstream and downstream velocities. By using complex-number notation, the mean velocity can be written as

$$V_m e^{i\lambda_m} = \frac{V_1 e^{i\lambda_1} + V_2 e^{i\lambda_2}}{2} \tag{4}$$

Also by using the continuity condition (equation (1)), equation (3) can be written:

$$\frac{i\Gamma}{2t} = \frac{V_1 e^{i\lambda_1} - V_2 e^{i\lambda_2}}{2} \tag{5}$$

Solution of equations (4) and (5) together gives

$$V_1 e^{i\lambda_1} = V_m e^{i\lambda_m} + i\frac{\Gamma}{2t} \tag{6}$$

$$V_2 e^{i\lambda_2} = V_m e^{i\lambda_m} - i\frac{\Gamma}{2t} \tag{7}$$

FLOW IN REGION BOUNDED BY CONCENTRIC CIRCLES

Derivation of equations for velocity and potential. - The velocities on the boundaries of an annular region, in which there are placed two complex sources, must be found. By means of successive reflections of the singularities in the two boundaries, an expression will be obtained for these velocities. The details of the process of reflection will be presented for only one of the singularities inasmuch as the process will be the same for both. In figure 3, A and C are, respectively, the radii of the inner and the outer circular boundaries of the annular region, which is so located that the centers of these circles are at the origin. The original singularity is located at B.

For convenience, the real (source) and the imaginary (vortex) parts of the singularity are considered separately. First to be considered is the reflection of the source located at B. The boundary condition on the surface of radius A is that there should be no component of velocity normal to this surface. In order that this condition be fulfilled, a like source must be added at a point inverse to B with respect to the circle A and a source of opposite sign must be added at the center of the circle. The boundary condition on surface C is the same as on surface A. It is apparent that the first reflected source now violates the boundary condition on the outer boundary. This image singularity must therefore be reflected in the outer boundary, which again necessitates a reflection in the inner boundary and so on indefinitely. If the reflection process is considered to begin with a reflection in the outer boundary, another series of images in different positions results. An additional singularity has to be placed at the center of the circle for each step of the reflection process. The total flow across boundaries A and C must be zero. The real parts of the two original singularities at B and B' are therefore equal in magnitude and opposite in sign. The reflection is carried out in the same manner for the other original singularity located at B'.

The reflection scheme for the vortex is similar to the one for the source except that on each reflection the sign of the reflected vortex is changed and the singularity added at the center of the circle has a sign opposite to that of the reflection. The magnitudes of the distance from the origin to the reflections are shown in figure 3.

Let

X source strength of singularity at B

-Y vortex strength of singularity at B

X' source strength of singularity at B'

-Y' vortex strength of singularity at B'

As previously discussed, the following relation must exist:

$$X = -X' \tag{8}$$

The potential at a point z due to a source of strength X located at a point z_0 is, if the potential is to be zero, at a point z_1,

$$\frac{X}{2\pi} \log_e \left(\frac{z-z_0}{z_1-z_0}\right)$$

For the vortex -Y located at z_0, the potential at z is, if it is desired that the potential be zero at a point z_1,

$$\frac{iY}{2\pi} \log_e \left(\frac{z-z_0}{z_1-z_0}\right)$$

The potential due to the source at B and its reflections can be written (see fig. 3 for locations of reflections)

$$W_X(z) = \frac{X}{2\pi} \left\{ \log_e \left(\frac{z-B}{A-B}\right) + \sum_1^{\infty} \log_e \left[\frac{A}{z} \frac{z - \frac{1}{\bar{B}} \frac{A^{2n}}{C^{2(n-1)}}}{A - \frac{1}{\bar{B}} \frac{A^{2n}}{C^{2(n-1)}}} \right] + \right.$$

$$\left. \sum_1^{\infty} \log_e \left[\frac{A}{z} \frac{z - B \frac{A^{2n}}{C^{2n}}}{A - B \frac{A^{2n}}{C^{2n}}} \right] + \sum_1^{\infty} \log_e \left[\frac{z - \frac{1}{\bar{B}} \frac{C^{2n}}{A^{2(n-1)}}}{A - \frac{1}{\bar{B}} \frac{C^{2n}}{A^{2(n-1)}}} \right] + \sum_1^{\infty} \log_e \left[\frac{z - B \frac{C^{2n}}{A^{2n}}}{A - B \frac{C^{2n}}{A^{2n}}} \right] \right\} \tag{9}$$

The point A was selected as the point for which the potential is zero. The potential due to the vortex at B and its reflections is

$$W_{iY}(z) = \frac{iY}{2\pi}\left\{\log_e\left(\frac{z-B}{A-B}\right) - \sum_{1}^{\infty}\log_e\left[\frac{A}{z}\,\frac{z-\frac{1}{\bar{B}}\,\frac{A^{2n}}{C^{2(n-1)}}}{A-\frac{1}{\bar{B}}\,\frac{A^{2n}}{C^{2(n-1)}}}\right] + \sum_{1}^{\infty}\log_e\left[\frac{z-B\frac{C^{2n}}{A^{2n}}}{A-B\frac{C^{2n}}{A^{2n}}}\right] + \right.$$

$$\left.\sum_{1}^{\infty}\log_e\left[\frac{A}{z}\,\frac{z-B\,\frac{A^{2n}}{C^{2n}}}{A-B\,\frac{A^{2n}}{C^{2n}}}\right] - \sum_{1}^{\infty}\log_e\left[\frac{z-\frac{1}{\bar{B}}\,\frac{C^{2n}}{A^{2(n-1)}}}{A-\frac{1}{\bar{B}}\,\frac{C^{2n}}{A^{2(n-1)}}}\right]\right\} \tag{10}$$

There are similar expressions for the complex potential that result from the source and the vortex at B'.

For the total potential due to both singularities

$$W(z) = W_{X+iY}(z) + W_{X'+iY'}(z) = \frac{X+iY}{2\pi}\left\{\log_e\left(\frac{z-B}{A-B}\right) + \sum_{1}^{\infty}\log_e\left[\frac{z-B\,\frac{C^{2n}}{A^{2n}}}{A-B\,\frac{C^{2n}}{A^{2n}}}\right] + \sum_{1}^{\infty}\log_e\left[\frac{A}{z}\,\frac{z-B\,\frac{A^{2n}}{C^{2n}}}{A-B\,\frac{A^{2n}}{C^{2n}}}\right]\right\} +$$

$$\frac{X'+iY'}{2\pi}\left\{\log_e\left(\frac{z-B'}{A-B'}\right) + \sum_{1}^{\infty}\log_e\left[\frac{z-B'\,\frac{C^{2n}}{A^{2n}}}{A-B'\,\frac{C^{2n}}{A^{2n}}}\right] + \sum_{1}^{\infty}\log_e\left[\frac{A}{z}\,\frac{z-B'\,\frac{A^{2n}}{C^{2n}}}{A-B'\,\frac{A^{2n}}{C^{2n}}}\right]\right\} + \frac{X-iY}{2\pi}\left\{\sum_{1}^{\infty}\log_e\left[\frac{A}{z}\,\frac{z-\frac{1}{\bar{B}}\,\frac{A^{2n}}{C^{2(n-1)}}}{A-\frac{1}{\bar{B}}\,\frac{A^{2n}}{C^{2(n-1)}}}\right] + \right.$$

$$\left.\sum_{1}^{\infty}\log_e\left[\frac{z-\frac{1}{\bar{B}}\,\frac{C^{2n}}{A^{2(n-1)}}}{A-\frac{1}{\bar{B}}\,\frac{C^{2n}}{A^{2(n-1)}}}\right]\right\} + \frac{X'-iY'}{2\pi}\left\{\sum_{1}^{\infty}\log_e\left[\frac{A}{z}\,\frac{z-\frac{1}{\bar{B}'}\,\frac{A^{2n}}{C^{2(n-1)}}}{A-\frac{1}{\bar{B}'}\,\frac{A^{2n}}{C^{2(n-1)}}}\right] + \sum_{1}^{\infty}\log_e\left[\frac{z-\frac{1}{\bar{B}'}\,\frac{C^{2n}}{A^{2(n-1)}}}{A-\frac{1}{\bar{B}'}\,\frac{C^{2n}}{A^{2(n-1)}}}\right]\right\} \tag{11}$$

These series are absolutely convergent in a closed region that excludes the singularities B and B'. Differentiation of equation (11) yields the conjugate velocity

$$\bar{V}(z) = \frac{X+iY}{2\pi}\left\{\frac{1}{z - B} + \sum_{1}^{\infty}\left[\frac{1}{z - B\frac{A^{2n}}{C^{2n}}} - \frac{1}{z}\right] + \sum_{1}^{\infty}\frac{1}{z - B\frac{C^{2n}}{A^{2n}}}\right\} +$$

$$\frac{X'+iY'}{2\pi}\left\{\frac{1}{z - B'} + \sum_{1}^{\infty}\left[\frac{1}{z - B'\frac{A^{2n}}{C^{2n}}} - \frac{1}{z}\right] + \sum_{1}^{\infty}\frac{1}{z - B'\frac{C^{2n}}{A^{2n}}}\right\} +$$

$$\frac{X-iY}{2\pi}\left\{\sum_{1}^{\infty}\left[\frac{1}{z - \frac{1}{\bar{B}}\frac{C^{2n}}{A^{2(n-1)}}}\right] + \sum_{1}^{\infty}\left[-\frac{1}{z} + \frac{1}{z - \frac{1}{\bar{B}}\frac{A^{2n}}{C^{2(n-1)}}}\right]\right\} +$$

$$\frac{X'-iY'}{2\pi}\left\{\sum_{1}^{\infty}\left[\frac{1}{z - \frac{1}{\bar{B}'}\frac{C^{2n}}{A^{2(n-1)}}}\right] + \sum_{1}^{\infty}\left[-\frac{1}{z} + \frac{1}{z - \frac{1}{\bar{B}'}\frac{A^{2n}}{C^{2(n-1)}}}\right]\right\} \qquad (12)$$

Velocity and potential in closed form. - In order to evaluate equations (11) and (12) more readily, the annular region in the z-plane is transformed into a rectangle in the w-plane. The transformation used is

$$z = e^{w}$$

By introducing the notation

$$\bar{V}(w) = \bar{V}[z(w)] = \bar{V}(e^{w})$$

equation (11) can be written as a function of w

$$\bar{V}(w) = e^{-w}\left[A_1 + \frac{K}{\omega_1}\left(\frac{X+iY}{2\pi}\left\{Z\left[\frac{K}{\omega_1}(w-b)\right] + \frac{\operatorname{cn}\left[\frac{K}{\omega_1}(w-b)\right]\operatorname{dn}\left[\frac{K}{\omega_1}(w-b)\right]}{\operatorname{sn}\left[\frac{K}{\omega_1}(w-b)\right]}\right\} + \right.\right.$$

$$\frac{X-iY}{2\pi}\left\{Z\left[\frac{K}{\omega_1}(w+\bar{b}+2a)\right] + \frac{\operatorname{cn}\left[\frac{K}{\omega_1}(w+\bar{b}+2a)\right]\operatorname{dn}\left[\frac{K}{\omega_1}(w+\bar{b}+2a)\right]}{\operatorname{sn}\left[\frac{K}{\omega_1}(w+\bar{b}+2a)\right]}\right\} +$$

$$\frac{X'+iY'}{2\pi}\left\{Z\left[\frac{K}{\omega_1}(w-b')\right] + \frac{\operatorname{cn}\left[\frac{K}{\omega_1}(w-b')\right]\operatorname{dn}\left[\frac{K}{\omega_1}(w-b')\right]}{\operatorname{sn}\left[\frac{K}{\omega_1}(w-b')\right]}\right\} +$$

$$\left.\left.\frac{X'-iY'}{2\pi}\left\{Z\left[\frac{K}{\omega_1}(w+\bar{b}'+2a)\right] + \frac{\operatorname{cn}\left[\frac{K}{\omega_1}(w+\bar{b}'+2a)\right]\operatorname{dn}\left[\frac{K}{\omega_1}(w+\bar{b}'+2a)\right]}{\operatorname{sn}\left[\frac{K}{\omega_1}(w+\bar{b}'+2a)\right]}\right\}\right)\right] \tag{13}$$

The potential can also be written in closed form as a function of w.

$$W(w) = W(e^{w}) = \frac{X+iY}{2\pi}\log_e \Theta\left[\frac{K}{\omega_1}(w-b) + iK'\right] +$$

$$\frac{X-iY}{2\pi}\log_e \Theta\left[\frac{K}{\omega_1}(w+\bar{b}+2a)+iK'\right] + \frac{X'+iY'}{2\pi}\log_e \Theta\left[\frac{K}{\omega_1}(w-b')+iK'\right] +$$

$$\frac{X'-iY'}{2\pi}\log_e \Theta\left[\frac{K}{\omega_1}(w+\bar{b}'+2a) + iK'\right] + A_1 w \tag{14}$$

where A_1 is an imaginary constant. Equations (13) and (14) are derived in appendix B. The symbols for elliptic functions are those defined in reference 4.

SOLUTION OF INVERSE PROBLEM

The inverse problem, as considered herein, is to find a pair of profile shapes that when suitably arranged in a tandem cascade will have a prescribed turning and surface velocity distribution. The region external to the tandem cascade is to be conformally transformed into a standard region through which the flow is known. The annular region between two concentric circles with suitably placed flow singularities is chosen as the standard region: (1) The general form of the mapping function together with certain conditions upon it will be found. (2) A method of computing the values of the mapping function along the boundaries of the annular region is given. (3) Because the assigned conditions may not give a closed profile, a method is given by means of which the initially assigned velocity distributions may be modified to give closed profiles.

Derivation of mapping function. - The σ-plane is the plane of the original tandem cascade. In order to transform the tandem cascade into the standard region, the following series of transformations is used:

(1)
$$\log_e \xi = \frac{2\pi\sigma}{t} \tag{15}$$

Because of the period properties of the exponential function, equation (15) transforms the tandem cascade into two closed figures in the ξ-plane (fig. 1(b)). Under this transformation, points for which $\mathrm{Re}(\sigma)$ is very large positively are transformed into points that are a large distance from the origin in the ξ-plane. Points having a very large real part that is negative are transformed into points near the origin. Therefore,

$$\lim_{\mathrm{Re}(\sigma) \to +\infty} \xi(\sigma) = \infty \tag{16}$$

$$\lim_{\mathrm{Re}(\sigma) \to -\infty} \xi(\sigma) = 0 \tag{17}$$

(2) According to reference 5, it is possible to transform conformally the region exterior to the closed figures in the ξ-plane into the area external to two circles. It is convenient to write this transformation in the following form:

$$\frac{d\xi}{d\zeta_1} = e^{f(\zeta_1)}$$

One circle, the correspondence of one point thereon, and in addition, the location of the center of the second circle can be specified. If the point $\zeta = \infty$ is transformed into some finite point of the ζ_1-plane, a bilinear transformation can be used to transform this finite point into the point at infinity in the ζ_2-plane. Under a bilinear transformation, circles are preserved as circles. A translation and a rotation can then be used to place the circles as shown in figure 1(c) where the point at infinity has been transformed into the point at infinity in this plane, the ζ-plane, and the centers of the circles lie on the axis of real numbers, equidistant from the axis of imaginary numbers. This entire series of transformations can be combined and written as

$$\frac{d\zeta}{d\zeta} = e^{f(\zeta)} \tag{18}$$

(3) Circles C and D of the ζ-plane can be transformed into concentric circles in the z-plane by

$$z = \frac{\zeta - k_1}{\zeta - k_2} \tag{19}$$

where k_1 and k_2 are points such that they are inverse points with respect to either circle. Therefore,

$$(k_1-k)\ (k_2-k) = R^2{}_C$$

$$(k_1+k)\ (k_2+k) = R^2{}_D$$

where R_C and R_D are the radii of the two circles in the ζ-plane. These two equations may be solved for k_1 and k_2.

If equation (19) is solved for ζ in terms of z,

$$\zeta = \frac{k_2 z + k_1}{z-1}$$

From which

$$\frac{d\zeta}{dz} = \frac{-(k_2+k_1)}{(z-1)^2}$$

and therefore,

$$\frac{d\zeta}{dz} = \frac{d\zeta}{d\xi}\frac{d\xi}{dz} = \frac{-e^{f(\zeta)}(k_2 + k_1)}{(z - 1)^2} \tag{20}$$

When the notation $f(z)$ is used for $f\left[\zeta(z)\right]$, equation (20) becomes

$$\frac{d\xi}{dz} = \frac{-e^{f(z)}(k_2 + k_1)}{(z - 1)^2} \tag{21}$$

If the point to which $\xi = 0$ is transformed is denoted by $Re^{i\gamma}$

$$\xi = \frac{z - Re^{i\gamma}}{z - 1} e^{F(z)} \tag{22}$$

where $F(z)$ is an entire function and

$$\frac{d\xi}{dz} = \frac{e^{F(z)}}{(z - 1)^2}\left[(z - Re^{i\gamma})(z - 1) F'(z) + (Re^{i\gamma} - 1)\right] \tag{23}$$

Equating these two expressions for $\frac{d\xi}{dz}$ and solving for $F'(z)$ gives

$$F'(z) = \frac{-(k_2 + k_1)\ e^{f(z) - F(z)} - (Re^{i\gamma} - 1)}{(z - Re^{i\gamma})(z - 1)}$$

This equation can be rewritten as

$$F'(z) = \frac{k_3\ (1 - e^{g(z)})}{(z - Re^{i\gamma})(z - 1)} \tag{24}$$

where

$$k_3 = 1 - Re^{i\gamma}$$

and

$$e^{g(z)} = \frac{(k_1 + k_2)\ e^{f(z)} - F(z)}{1 - Re^{i\gamma}} \tag{25}$$

Because $F(z)$ and therefore $F'(z)$ are regular in the annular region, $1 - e^{g(z)}$ must have zeros at 1 and $Re^{i\gamma}$. Therefore

$$g(1) = 0 \tag{26}$$

$$g(Re^{i\gamma}) = 0 \tag{27}$$

Because

$$\log_e \xi = \frac{2\pi\sigma}{t} \tag{15}$$

therefore,

$$\frac{d\sigma}{d\xi} = \frac{t}{2\pi} e^{-\frac{2\pi\sigma}{t}} \tag{28}$$

Therefore, by using equations (22) to (25) and (28)

$$\frac{d\sigma}{dz} = \frac{d\sigma}{d\xi}\frac{d\xi}{dz} = \frac{-tk_3\ e^{g(z)}}{2\pi(z - 1)(z - Re^{i\gamma})} \tag{29}$$

The conditions on $g(z)$ may be more easily applied if the function $Q(z)$ is introduced such that

$$g(z) = Q(z)\ h(z) \tag{30}$$

where $h(z)$ is analytic in the annular region and $Q(1)$ and $Q(Re^{i\gamma})$ are both zero. In addition, $Q(z)$ must be real on both boundaries. An example of a function suitable for Q is presented in appendix C.

Method of computation of mapping function. - The σ-plane and the z-plane are related by a conformal transformation

$$U(\sigma) = W(z)$$

where

$U(\sigma)$ potential function in tandem-cascade plane

$W(z)$ potential function in annular-region plane

Differentiation of the preceding equation yields:

$$\frac{dU(\sigma)}{d\sigma}\frac{d\sigma}{dz}=\frac{dW(z)}{dz}$$

or

$$\frac{d\sigma}{dz}=\frac{\overline{V}(z)}{\overline{u}(\sigma)} \tag{31}$$

Let $u(z)=\overline{u}(\sigma(z))$.

Equation (31) can be rewritten as

$$\frac{d\sigma}{dz}=\frac{\overline{V}(z)}{u(z)} \tag{31a}$$

From equation (29)

$$\frac{d\sigma}{dz}=\frac{-t}{2\pi}e^{g(z)}\left(\frac{1}{z-1}-\frac{1}{z-Re^{i\Upsilon}}\right) \tag{32}$$

From equations (16) and (17) and the discussion preceding equation (18),

$$\lim_{Re(\sigma)\to+\infty} z=1$$

$$\lim_{Re(\sigma)\to-\infty} z=Re^{i\Upsilon}$$

From equations (6), (7), and (31a),

$$\lim_{z\to 1} u(z)=V_m e^{i\lambda_m}+\frac{i\Gamma}{2t}$$

$$\lim_{z\to Re^{i\Upsilon}} u(z)=V_m e^{i\lambda_m}-\frac{i\Gamma}{2t}$$

so that the residues of $\bar{V}(z)$ at 1 and $Re^{i\gamma}$ are, respectively,

$$t\left(-V_m e^{i\lambda_m} - \frac{i\Gamma}{2t}\right)$$

$$t\left(V_m e^{i\lambda_m} - \frac{i\Gamma}{2t}\right)$$

If the points 1 and $Re^{i\gamma}$ are identified with the points B and B' of equations (11) and (12), the source and vortex strengths are

$$X = -tV_m \cos \lambda_m$$

$$Y = \frac{-\Gamma}{2} + tV_m \sin \lambda_m$$

$$X' = tV_m \cos \lambda_m$$

$$Y' = \frac{-\Gamma}{2} - tV_m \sin \lambda_m$$

Next, an annular region such that the ranges of potential are correctly matched to the assigned ranges of potential on the two blade shapes must be found. The following scheme may be used: Let $Re^{i\gamma}$ equal -1. The radii of the inner and outer boundaries are left to be determined in a manner such that the ranges of potential have the proper values.

A certain amount of control over the relative positions of the two cascades can be achieved by specifying the difference in the stream function for the two blades and the potential difference between the tail stagnation point on one blade and the tail stagnation point on the other blade. These additional specifications increase the difficulty of determining the boundary radii and the location of the point $Re^{i\gamma}$.

Once the radii of the boundaries and the locations of the singularities have been determined, the correspondence between points on the two blades and points on the circular boundary may be found by matching points having the same potential (as in reference 6). From this correspondence is known the assigned blade velocity as a function of the central angle on the corresponding circle.

From equations (29) and (30)

$$\frac{d\sigma}{dz} = \frac{-tk_3 \; e^{Q(z)}h(z)}{2\pi \; (z - 1)(z - Re^{i\Upsilon})} \tag{34}$$

where

$$h(z) = \sum_{-\infty}^{+\infty} (a_n + ib_n)z^n$$

$$= p(x,y) + iq(x,y) \tag{35}$$

$$\frac{\overline{V}(z)}{u(z)} = \frac{d\sigma}{dz} \tag{31a}$$

On the boundaries of the annular region, $Q(z)$ is real and

$$\left|\frac{\overline{V}(z)}{u(z)}\right| = \left|\frac{1}{z - 1}\right| \left|\frac{1}{z - Re^{i\Upsilon}}\right| \left|e^{Q(z)}h(z)\right| \left|\frac{-tk_3}{2\pi}\right| \tag{36}$$

Therefore, on either circular boundary

$$\operatorname{Re} h(z) = \frac{1}{Q(z)} \left[\log_e \left|\frac{\overline{V}(z)}{u(z)}\right| + \log_e \left|z-1\right| + \log_e \left|z - Re^{i\Upsilon}\right| - \log_e \left|\frac{-tk_3}{2\pi}\right|\right] \tag{37}$$

By use of equation (37), Re (h(z)) can be computed on both circular boundaries. In order to compute the function conjugate to Re (h(z)), use is made of Villat's analogue, for the annular region, of Poisson's integral. These equations are derived in reference 7. A form more convenient for the present purpose is given in reference 3 (p. 13, equations (23) and (24)).

$$q_A(\varphi') = b_o + \frac{1}{\pi}\int_0^{2\pi} p_C(\varphi)\, Z\,(\varphi - \varphi')\, d\varphi -$$

$$\frac{1}{\pi}\int_0^{2\pi} p_A(\varphi)\, Z_1\,(\varphi - \varphi')\, d\varphi \tag{38}$$

$$q_C(\varphi') = b_o + \frac{1}{\pi}\int_0^{2\pi} p_C(\varphi)\, Z_1\,(\varphi - \varphi')\, d\varphi$$

$$- \frac{1}{\pi}\int_0^{2\pi} p_A(\varphi)\, Z\,(\varphi - \varphi')\, d\varphi \tag{39}$$

where

$p_A(\varphi)$ real part of $h(z)$ on inner boundary as function of central angle φ

$p_C(\varphi)$ real part of $h(z)$ on outer boundary as function of central angle φ

$q_A(\varphi)$ imaginary part of $h(z)$ on inner boundary as function of central angle φ

$q_C(\varphi)$ imaginary part of $h(z)$ on outer boundary as function of central angle φ

b_o constant, which may be taken as zero

$Z_1(u)$ $\frac{H'(u)}{H(u)}$ (See reference 4 for definition of the H function.)

From equation (34)

$$d\sigma = \frac{-tk_3\ e^{Q(z)}\left[p(x,y) + iq(x,y)\right]}{2\pi\ (z - 1)(z - Re^{i\gamma})}\, dz \tag{40}$$

Integration of equation (40) gives the required value of σ.

In order to find the position of one cascade relative to the other, it is necessary to integrate $\frac{d\sigma}{dz}$ along a line from one circular boundary, across the annular region to the other circular boundary. This procedure involves knowing the values of $h(z)$ along this line. These values can be found by means of the following equation, which (with different notation) is found in references 6 and 3:

$$h(z) = (a_o + ib_o) - \frac{i}{\pi}\int_0^{2\pi} p_C(\varphi)\ Z\left[i\ \log\left(\frac{z}{A}\right) + \varphi\right] d\varphi +$$

$$\frac{i}{\pi}\int_0^{2\pi} p_A(\varphi)\ Z\left[i\ \log\left(\frac{z}{C}\right) + \varphi\right] d\varphi \tag{41}$$

where

$$a_o = \frac{1}{\pi}\int_0^{2\pi} p_A(\varphi)\ d\varphi = \frac{1}{\pi}\int_0^{2\pi} p_C(\varphi)\ d\varphi$$

<u>Modification of assigned velocity to obtain closed profiles.</u> - The preceding development does not guarantee that the resulting blade profiles will be closed curves. In order to modify the initially assigned velocity so that closed blade shapes will result, the following procedure is used:

The sum of the residues of the derivative of the mapping function inside the contour indicated in figure 4(a) is zero; that is, the integral around the contour will be zero. The contour in the σ-plane corresponding to the contour indicated in figure 4(a) therefore will be closed, whereas the portions of the contour corresponding to the bounding circles might not be closed curves. If this part of the contour is open, the opening for one blade will be the negative of the opening for the second blade because the integration around the outer circle is in the negative direction (fig. 4(a)).

The amount by which the profiles fail to close can be reduced to zero by adding to the derivative of the mapping function, a function having a pole at the origin such that the residue of this function at its pole is equal to the negative of the amount of opening divided by 2π.

If the amount of lack of closure is equal to m_o, the modified mapping-function derivative can be written as

$$\frac{d\sigma'}{dz} = \frac{d\sigma}{dz} - \frac{m_o}{2\pi z} = \frac{\overline{V}'(z)}{u'(z)}$$

Because the velocity remains undisturbed in the annular region

$$\overline{V}'(z) = \overline{V}(z)$$

Therefore, the altered velocity is

$$u'(z) = \frac{z\ u(z)\ \overline{V}(z)}{z\ \overline{V}(z) - m_o\ u(z)} \tag{42}$$

and the altered arc length is given by

$$d\sigma' = \left[- \frac{m_o}{2\pi z} - \frac{\overline{V}(z)}{u(z)}\right] dz \tag{43}$$

SOLUTION OF DIRECT PROBLEM

The direct problem is that of finding the flow of fluid past blades of known shape. In addition to the blade profiles, the upstream and downstream velocities are also known. The Kutta condition of finite velocities at the trailing edge of the blades is used to determine the blade circulations.

As in the section "Solution of Inverse Problem", by the application of an exponential transformation, the flow through the tandem cascade is transformed into the flow about two closed profiles with a flow singularity in the finite part of the plane; this is essentially the problem of the two-dimensional biplane. A method by which the transformation from the region external to the two closed shapes to the region between two concentric circles can be computed by means of successive approximations to the solutions of a pair of integral equations is given in reference 3.

In the biplane problem, the flow singularity is a single dipole. In the problem under discussion, there are two singularities in the flow field. In order to determine the location of the addition singularity, use must be made of a trial-and-error procedure; an equation similar to equation (41) is used to determine the values of the mapping function at points interior to the annular region. From equations (13) and (14), the velocity and the potential on the annular region boundary is known. The analysis is completed by transforming these velocities back to the original tandem cascade configuration.

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, February 9, 1951.

APPENDIX A

MAIN SYMBOLS

A, C	radii of boundaries in z-plane
a, c	$a \equiv -\log_e A$ $\quad c \equiv \log_e C$
A_1, k, k_1, k_2, k_3	complex constants
B, B'	singularity positions in z-plane
b, b'	$b \equiv \log_e B$ $\quad b' \equiv \log_e B'$
$f(\zeta)$	analytic function of ζ
$f(z), F(z), g(z),$ $h(z), Q(z)$	analytic functions of z
$G(w)$	velocity function in w-plane
$h_1(w), h_2(w),$ $h_3(w), h_4(w)$	arbitrary entire functions
m_o	residue of function added to achieve closure
$p(x,y)$	Re $[h(z)]$ where $z = x + iy$
$p_A(\varphi)$	Re $[h(z)]$ as function of φ along $\lvert z\rvert = A$, $z = Ae^{i\varphi}$
$p_C(\varphi)$	Re $[h(z)]$ as function of φ along $\lvert z\rvert = C$, $z = Ce^{i\varphi}$
$q(x,y)$	Im $[h(z)]$ where $z = x + iy$
$q_A(\varphi)$	Im $[h(z)]$ as function of φ along $\lvert z\rvert = A$, $z = Ae^{i\varphi}$
$q_B(\varphi)$	Im $[h(z)]$ as function of φ along $\lvert z\rvert = C$, $z = Ce^{i\varphi}$
$Re^{i\gamma}$	singularity position in z-plane ($Re^{i\gamma} = B'$)
r	$\log_e R$
t	spacing
$U(\sigma), W(z)$	potential functions

$\overline{V}(z), \overline{u}(\sigma), u$	velocity functions
$V(w)$	$V(w) \equiv V[z(w)]$
$u(w)$	$u(w) \equiv \overline{u}[\sigma(w)]$
$V_1 e^{i\lambda_1}$	upstream velocity
$V_2 e^{i\lambda_2}$	downstream velocity
$V_m e^{i\lambda_m}$	vector mean velocity
X, X'	source strengths
$-Y, -Y'$	vortex strengths
Γ	circulation
σ-plane	tandem-cascade plane
σ'-plane	modified tandem-cascade plane
z-plane	annular-region plane
$\xi, \zeta, \zeta_1, \zeta_2, w$	auxiliary variables used in transformation

Notation used for elliptic functions is that found in reference 4 except for $Z_1(u)$, which is defined in text following equation (39).

APPENDIX B

EXPRESSION OF VELOCITY AND POTENTIAL IN TERMS OF ELLIPTIC FUNCTIONS

Velocity expressed in closed form. - In order that equation (12) may be more readily evaluated, the annular region in the z-plane is transformed into a rectangle in the w-plane. The transformation used is

$$z = e^{w} \tag{B1}$$

Let

$$\overline{V}(w) = \overline{V}[z(w)] = \overline{V}(e^{w})$$

Then equation (12) can be rewritten as a function of w:

$$\overline{V}(w) = \frac{X+iY}{2\pi}\left\{\frac{1}{e^{w} - B} + \sum_{1}^{\infty}\left[\frac{1}{e^{w} - B\dfrac{A^{2n}}{C^{2n}}} - \frac{1}{e^{w}}\right] + \sum_{1}^{\infty}\frac{1}{e^{w} - B\dfrac{C^{2n}}{A^{2n}}}\right\} +$$

$$\frac{X-iY}{2\pi}\left\{\sum_{1}^{\infty}\left[\frac{1}{e^{w} - \dfrac{1}{\overline{B}}\dfrac{A^{2n}}{C^{2(n-1)}}} - \frac{1}{e^{w}}\right] + \sum_{1}^{\infty}\frac{1}{e^{w} - \dfrac{1}{\overline{B}}\dfrac{C^{2n}}{A^{2(n-1)}}}\right\} +$$

$$\frac{X'+iY'}{2\pi}\left\{\frac{1}{e^{w} - B'} + \sum_{1}^{\infty}\left[\frac{1}{e^{w} - B'\dfrac{A^{2n}}{C^{2n}}} - \frac{1}{e^{w}}\right] + \sum_{1}^{\infty}\frac{1}{e^{w} - B'\dfrac{C^{2n}}{A^{2n}}}\right\} +$$

$$\frac{X'-iY'}{2\pi}\left\{\sum_{1}^{\infty}\left[\frac{1}{e^{w} - \dfrac{1}{\overline{B'}}\dfrac{A^{2n}}{C^{2(n-1)}}} - \frac{1}{e^{w}}\right] + \sum_{1}^{\infty}\frac{1}{e^{w} - \dfrac{1}{\overline{B'}}\dfrac{C^{2n}}{A^{2(n-1)}}}\right\} \tag{B2}$$

Let

$$\log_e A = -a$$

$$\log_e B = b$$

$$\log_e C = c$$

$$\log_e B' = b'$$

Further let,

$$-\pi \leq \mathrm{Im}(w) \leq \pi$$

$$-a \leq \mathrm{Re}(w) \leq c$$

Then, because $X = -X'$, for w contained within this rectangle (fig. 1(e))

$$\overline{V}(w+2a+2c) = \frac{A^2}{C^2}\,\overline{V}(w) \tag{B3}$$

because

$$e^{w+2\pi i} = e^w$$

$$\overline{V}(w+2\pi i) = \overline{V}(w)$$

If

$$G(w) = e^w\,\overline{V}(w) \tag{B4}$$

Then

$$G(w+2\pi i) = G(w) \tag{B5a}$$

and

$$G(w+2a+2c) = e^{w+2a+2c}\,\overline{V}(w+2a+2c)$$

$$= e^w\,\overline{V}(w)\,\frac{A^2}{C^2}\,\frac{C^2}{A^2} = G(w) \tag{B5b}$$

Therefore $G(w)$ is a doubly periodic or elliptic function and has a real period $(2a+2c)$ plus an imaginary period $2\pi i$.

Consider one of the terms of $G(w)$

$$e^w\left(\frac{1}{e^w - B\frac{A^{2n}}{C^{2n}}} - \frac{1}{e^w}\right) \tag{B6}$$

This function has an expansion in terms of $\left[w - b + 2n(a + c)\right]$

$$-1 + \frac{e^w}{e^w - B\frac{A^{2n}}{C^{2n}}} = -1 + \left(\frac{1}{w - b + 2n\,(a + c)}\left\{\left[\sum_{k=0}^{\infty}\frac{\left[w - b + 2n\,(a + c)\right]^k}{k!}\right]\right\}\right.$$

$$\left.\left\{\left[\sum_{k=0}^{\infty}\frac{B_k}{k!}\left[w - b + 2n\,(a + c)\right]^k\right]\right\}\right) = \left[\frac{1}{w - b + 2n\,(a + c)} + \ . \ . \ .\right] -1 \tag{B7}$$

where the B_k are Bernoulli numbers (reference 8, p. 183). The first Bernoulli number B_0 is equal to 1. The residue of the function of equation (B6) is therefore 1. Similarly, the residues of the functions $\frac{e^w}{e^w - B}$, $\frac{e^w}{e^w - B\frac{C^{2n}}{A^{2n}}}$, and $\frac{e^w}{e^w - \frac{1}{B}\frac{A^{2n}}{C^{2(n-1)}}} - 1$, and so forth are each 1.

Consider a parallelogram in the w-plane,

$$-i\pi \leq \mathrm{Im}(w) \leq \pi$$

$$\mathrm{Re}\ b + 2a \leq \mathrm{Re}(w) \leq \mathrm{Re}\ b + 2c$$

In this rectangle $G(w)$ has four poles located at b, $-\overline{b} - 2a$, b', and $-\overline{b}' - 2a$. The sum of the residues of $G(w)$ at these poles is

$$\frac{1}{2\pi}\left(X + iY + X - iY + X' + iY' + X' - iY'\right) = 0 \tag{B8}$$

because

$$X = -X'.$$

The principal part of $G(w)$ at each of these poles is, respectively,

$$\frac{1}{2\pi}\frac{X + iY}{w - b}$$

$$\frac{1}{2\pi}\frac{X - iY}{w + \bar{b} + 2a}$$

$$\frac{1}{2\pi}\frac{X' + iY'}{w - b'}$$

$$\frac{1}{2\pi}\frac{X' - iY'}{w + \bar{b}' + 2a}$$

According to reference 4 (p. 474), if

$$\sum_{m=1}^{m_r} A_{r,m} (z - \beta_r)^{-m}$$

is the principal part of an elliptic function $f(z)$ at its pole β_r, then

$$f(z) = A_1 + \sum_{r=1}^{n}\left[\sum_{m=1}^{m_r} \frac{(-1)^{m-1}}{(m - 1)!} A_{r,m} \frac{d^m}{dz^m} \log_e \vartheta_1 \left(\frac{\pi z - \pi\beta_r}{2\omega_1} \middle| \frac{\omega_2}{\omega_1}\right)\right] \tag{B9}$$

where A_1 is a constant.

From reference 4 (p. 479),

$$H(z) \equiv \vartheta_1 \left(z \vartheta_3^{-2} \middle| \frac{\omega_2}{\omega_1}\right)$$

$$\vartheta_3^{-2} = \frac{\pi}{2K}$$

so that, by using the older notation of Jacobi,

$$f(z) = A_1 + \sum_{r=1}^{n}\left\{\sum_{m=1}^{m_r} \frac{(-1)^{m-1}}{(m - 1)!} A_{r,m} \frac{d^m}{dz^m} \log_e H \left[\frac{K}{\omega_1} (z - \beta_r) \middle| \frac{iK'}{K}\right]\right\} \tag{B10}$$

For the present case,

$$A_{1,1} = \frac{X+iY}{2\pi}$$

$$A_{2,1} = \frac{X-iY}{2\pi}$$

$$A_{3,1} = \frac{X'+iY'}{2\pi}$$

$$A_{4,1} = \frac{X'-iY'}{2\pi}$$

$$\beta_1 = b$$

$$\beta_2 = -\bar{b} - 2a$$

$$\beta_3 = b'$$

$$\beta_4 = -\bar{b}' - 2a$$

so that,

$$G(w) = A_1 + \frac{K}{\omega_1}\left\{\left(\frac{X+iY}{2\pi}\right)\frac{H'\left[\frac{K}{\omega_1}(w-b)\right]}{H\left[\frac{K}{\omega_1}(w-b)\right]} + \left(\frac{X-iY}{2\pi}\right)\frac{H'\left[\frac{K}{\omega_1}(w+\bar{b}+2a)\right]}{H\left[\frac{K}{\omega_1}(w+\bar{b}+2a)\right]} + \right.$$

$$\left.\left(\frac{X'+iY'}{2\pi}\right)\frac{H'\left[\frac{K}{\omega_1}(w-b')\right]}{H\left[\frac{K}{\omega_1}(w-b')\right]} + \left(\frac{X'-iY'}{2\pi}\right)\frac{H'\left[\frac{K}{\omega_1}(w+\bar{b}'+2a)\right]}{H\left[\frac{K}{\omega_1}(w+\bar{b}'+2a)\right]}\right\} \tag{B11}$$

or

$$G(w) = A_1 + \frac{K}{\omega_1}\left\{\left(\frac{X+iY}{2\pi}\right) Z\left[\frac{K}{\omega_1}(w-b)+iK'\right] + \left(\frac{X-iY}{2\pi}\right) Z\left[\frac{K}{\omega_1}(w+\bar{b}+2a)+iK'\right] + \right.$$

$$\left.\left(\frac{X'+iY'}{2\pi}\right) Z\left[\frac{K}{\omega_1}(w-b')+iK'\right] + \left(\frac{X'-iY'}{2\pi}\right) Z\left[\frac{K}{\omega_1}(w+\bar{b}'+2a)+iK'\right]\right\} \tag{B12}$$

where

$$Z(w) = \frac{-i\pi}{2k} + \frac{H'(w + iK')}{H(w + iK')}$$

When Jacobian elliptic functions are introduced into equation (B12) and equation (B4) is used in conjunction,

$$\begin{aligned}
\overline{V}(w) = e^{-w}G(w) = e^{-w}\Bigg[A_1 + \frac{K}{\omega_1}\Bigg(&\frac{X+iY}{2\pi}\left\{Z\left[\frac{K}{\omega_1}(w-b)\right] + \frac{\operatorname{cn}\left[\frac{K}{\omega_1}(w-b)\right]\operatorname{dn}\left[\frac{K}{\omega_1}(w-b)\right]}{\operatorname{sn}\left[\frac{K}{\omega_1}(w-b)\right]}\right\} + \\
&\frac{X-iY}{2\pi}\left\{Z\left[\frac{K}{\omega_1}(w + \overline{b} + 2a)\right] + \frac{\operatorname{cn}\left[\frac{K}{\omega_1}(w + \overline{b} + 2a)\right]\operatorname{dn}\left[\frac{K}{\omega_1}(w + \overline{b} + 2a)\right]}{\operatorname{sn}\left[\frac{K}{\omega_1}(w + \overline{b} + 2a)\right]}\right\} + \\
&\frac{X'+iY'}{2\pi}\left\{Z\left[\frac{K}{\omega_1}(w - b')\right] + \frac{\operatorname{cn}\left[\frac{K}{\omega_1}(w - b')\right]\operatorname{dn}\left[\frac{K}{\omega_1}(w - b')\right]}{\operatorname{sn}\left[\frac{K}{\omega_1}(w - b')\right]}\right\} + \\
&\frac{X'-iY'}{2\pi}\left\{Z\left[\frac{K}{\omega_1}(w + \overline{b}' + 2a)\right] + \frac{\operatorname{cn}\left[\frac{K}{\omega_1}(w + \overline{b}' + 2a)\right]\operatorname{dn}\left[\frac{K}{\omega_1}(w + \overline{b}' + 2a)\right]}{\operatorname{sn}\left[\frac{K}{\omega_1}(w + \overline{b}' + 2a)\right]}\right\}\Bigg)\Bigg]
\end{aligned} \tag{13}$$

Let

$$w = -a$$

$$\bar{V}(-a) = e^{a}\left[A_1 + \frac{K}{\omega_1}\left(\frac{X+iY}{2\pi}\left\{Z\left[\frac{K}{\omega_1}(-a-b)\right] + \frac{\text{cn}\left[\frac{K}{\omega_1}(-a-b)\right]\ \text{dn}\left[\frac{K}{\omega_1}(-a-b)\right]}{\text{sn}\left[\frac{K}{\omega_1}(-a-b)\right]}\right\} + \right.\right.$$

$$\frac{X-iY}{2\pi}\left\{Z\left[\frac{K}{\omega_1}(a+\bar{b})\right] + \frac{\text{cn}\left[\frac{K}{\omega_1}(a+\bar{b})\right]\ \text{dn}\left[\frac{K}{\omega_1}(a+\bar{b})\right]}{\text{sn}\left[\frac{K}{\omega_1}(a+\bar{b})\right]}\right\} +$$

$$\frac{X'+iY'}{2\pi}\left\{Z\left[\frac{K}{\omega_1}(-a-b')\right] + \frac{\text{cn}\left[\frac{K}{\omega_1}(-a-b')\right]\ \text{dn}\left[\frac{K}{\omega_1}(-a-b')\right]}{\text{sn}\left[\frac{K}{\omega_1}(-a-b')\right]}\right\} +$$

$$\left.\left.\frac{X'-iY'}{2\pi}\left\{Z\left[\frac{K}{\omega_1}(a+\bar{b}')\right] + \frac{\text{cn}\left[\frac{K}{\omega_1}(a+\bar{b}')\right]\ \text{dn}\left[\frac{K}{\omega_1}(a+\bar{b}')\right]}{\text{sn}\left[\frac{K}{\omega_1}(a+\bar{b}')\right]}\right\}\right)\right] \tag{B13}$$

For convenience let

$$\frac{K}{\omega_1}\,\text{Im}\ b + K' = v$$

$$\frac{K}{\omega_1}\,\text{Im}\ b' + K' = v'$$

$$\frac{K}{\omega_1}\,\text{Re}\ b + a = u$$

$$\frac{K}{\omega_1}\,\text{Re}\ b' + a = u'$$

Then equation (B13) can be rewritten as

$$\overline{V}(-a) = e^{a}\left(A_1 + \frac{2K}{\omega_1}\left\{X\left[Z(iv) + \frac{k^2\ \mathrm{sn}(iv)\ \mathrm{sn}^2(u)\ \mathrm{cn}(iv)\ \mathrm{dn}(iv)}{1 - k^2\ \mathrm{sn}^2 u\ \mathrm{sn}^2(iv)}\right] +\right.\right.$$

$$iY\left[Z(u) - \frac{k^2\ \mathrm{sn}^2(iv)\ \mathrm{sn}(u)\ \mathrm{cn}(u)\ \mathrm{dn}(u)}{1 - k^2\ \mathrm{sn}^2(u)\ \mathrm{sn}^2\ (iv)}\right] +$$

$$X'\left[Z(iv) + \frac{k^2\ \mathrm{sn}(iv')\ \mathrm{sn}^2(u')\ \mathrm{cn}(iv')\ \mathrm{dn}(iv')}{1 - k^2\ \mathrm{sn}^2(iv')\ \mathrm{sn}^2(u')}\right] +$$

$$\left.\left.Y'\left[Z(u') - \frac{k^2\ \mathrm{sn}(iv')\ \mathrm{sn}(u')\ \mathrm{cn}(u')\ \mathrm{dn}(u')}{1 - k^2\ \mathrm{sn}^2(u')\ \mathrm{sn}^2(iv')}\right]\right\}\right) \qquad \text{(B14)}$$

The point $w = -a$ corresponds in the z-plane to the point $z = A$. At this point, the velocity, in order to be tangent to the boundary, cannot have a real component. The coefficient of $\frac{2K}{\omega_1}$ in equation (B14) is an imaginary number. Therefore, A_1 must be an imaginary number.

$$\mathrm{Re}(A_1) = 0 \qquad \text{(B15)}$$

$$\frac{dW(z)}{dz} = \overline{V}(z) \qquad \text{(B16)}$$

$$\frac{dW\left[z(w)\right]}{dw}\ \frac{dw}{dz} = \overline{V}(z) \qquad \text{(B17)}$$

From equation (B1)

$$\frac{dz}{dw} = e^{w} \qquad \text{(B18)}$$

If in equation (11), z is replaced by e^w, equation (11) is equivalent to

$$
e^{W(w)} = \left\{\prod_{1}^{\infty}\frac{A}{e^w}\left[\frac{e^w - B\left(\frac{A}{C}\right)^{2n}}{A - B\left(\frac{A}{C}\right)^{2n}}\right]\right\}^{\frac{X+iY}{2\pi}}\left\{\prod_{0}^{\infty}\left[\frac{e^w - B\left(\frac{C}{A}\right)^{2n}}{A - B\left(\frac{C}{A}\right)^{2n}}\right]\right\}^{\frac{X+iY}{2\pi}}
$$

$$
\left\{\prod_{1}^{\infty}\frac{A}{e^w}\left[\frac{e^w - B'\left(\frac{A}{C}\right)^{2n}}{A - B'\left(\frac{A}{C}\right)^{2n}}\right]\right\}^{\frac{X'+iY'}{2\pi}}\left\{\prod_{0}^{\infty}\left[\frac{e^w - B'\left(\frac{C}{A}\right)^{2n}}{A - B'\left(\frac{C}{A}\right)^{2n}}\right]\right\}^{\frac{X'+iY'}{2\pi}}
$$

$$
\left\{\prod_{1}^{\infty}\frac{A}{e^w}\left[\frac{e^w - \frac{1}{\bar{\bar{B}}}\,\frac{A^{2n}}{C^{2(n-1)}}}{A - \frac{1}{\bar{\bar{B}}}\,\frac{A^{2n}}{C^{2(n-1)}}}\right]\right\}^{\frac{X-iY}{2\pi}}\left\{\prod_{1}^{\infty}\left[\frac{e^w - \frac{1}{\bar{\bar{B}}}\,\frac{C^{2n}}{A^{2(n-1)}}}{A - \frac{1}{\bar{\bar{B}}}\,\frac{C^{2n}}{A^{2(n-1)}}}\right]\right\}^{\frac{X-iY}{2\pi}}
$$

$$
\left\{\prod_{1}^{\infty}\frac{A}{e^w}\left[\frac{e^w - \frac{1}{\bar{\bar{B}}'}\,\frac{A^{2n}}{C^{2(n-1)}}}{A - \frac{1}{\bar{\bar{B}}'}\,\frac{A^{2n}}{C^{2(n-1)}}}\right]\right\}^{\frac{X'-iY'}{2\pi}}\left\{\prod_{1}^{\infty}\left[\frac{e^w - \frac{1}{\bar{\bar{B}}'}\,\frac{C^{2n}}{A^{2(n-1)}}}{A - \frac{1}{\bar{\bar{B}}'}\,\frac{C^{2n}}{A^{2(n-1)}}}\right]\right\}^{\frac{X'-iY'}{2\pi}} \qquad (B19)
$$

Consider the first bracketed term of equation (B19). This product has zeros of the order of 1 at points congruent to b modulo $2(j(a + c) + k\pi i)$, where j and k are integers. The function $\Theta\left[\frac{K}{\omega_1}(w - b) + iK'\right]$ has zeros at the same points. Hence, according to Weierstrass' factor theorem, the first bracketed term is equal to the following expression:

$$\left\{\Theta\left[\frac{K}{\omega_1}(w - b) + iK'\right] e^{h_1(w)}\right\}^{\frac{X+iY}{2\pi}}$$

where $h_1(w)$ is an arbitrary entire function. When the other bracketed terms are treated in a similar manner

$$e^{W(w)} = \left\{\Theta\left[\frac{K}{\omega_1}(w - b) + iK'\right] e^{h_1(w)}\right\}^{\frac{X+iY}{2\pi}} \left\{\Theta\left[\frac{K}{\omega_1}(w - b') + iK'\right] e^{h_2(w)}\right\}^{\frac{X'+iY'}{2\pi}}$$

$$\left\{\Theta\left[\frac{K}{\omega_1}(w + \bar{b} + 2a) + iK'\right] e^{h_3(w)}\right\}^{\frac{X-iY}{2\pi}} \left\{\Theta\left[\frac{K}{\omega_1}(w + \bar{b}' + 2a) + iK'\right] e^{h_4(w)}\right\}^{\frac{X'-iY'}{2\pi}} \tag{B20}$$

From equation (B20) it follows that

$$\begin{aligned} W(w) = {} & \frac{X+iY}{2\pi}\left\{\log_e \Theta\left[\frac{K}{\omega_1}(w - b) + iK'\right] + h_1(w)\right\} + \\ & \frac{X'+iY'}{2\pi}\left\{\log_e \Theta\left[\frac{K}{\omega_1}(w - b') + iK'\right] + h_2(w)\right\} + \\ & \frac{X-iY}{2\pi}\left\{\log_e \Theta\left[\frac{K}{\omega_1}(w + \bar{b} + 2a) + iK'\right] + h_3(w)\right\} + \\ & \frac{X'-iY'}{2\pi}\left\{\log_e \Theta\left[\frac{K}{\omega_1}(w + \bar{b}' + 2a) + iK'\right] + h_4(w)\right\} \end{aligned} \tag{B21}$$

It is now desirable to find the relation among the four arbitrary functions $h_1(w)$, $h_2(w)$, $h_3(w)$, $h_4(w)$, and the constant A_1. Differentiation of equation (B21) and use of equations (B17) and (B18) give

$$\overline{V}(w) = e^{-w}\left\{\frac{K}{\omega_1}\left(\frac{X+iY}{2\pi}\right)\frac{\Theta'\left[\frac{K}{\omega_1}(w-b)\ +iK'\right]}{\Theta\left[\frac{K}{\omega_1}(w-b)\ +iK'\right]} + \frac{K}{\omega_1}\left(\frac{X'+iY'}{2\pi}\right)\frac{\Theta'\left[\frac{K}{\omega_1}(w-b')\ +iK'\right]}{\Theta\left[\frac{K}{\omega_1}(w-b')\ +iK'\right]} + \right.$$

$$\frac{K}{\omega_1}\left(\frac{X'-iY'}{2\pi}\right)\frac{\Theta'\left[\frac{K}{\omega_1}(w+\overline{b}'+2a)\ +iK'\right]}{\Theta\left[\frac{K}{\omega_1}(w+\overline{b}'+2a)\ +iK'\right]} + \frac{K}{\omega_1}\left(\frac{X-iY}{2\pi}\right)\frac{\Theta'\left[\frac{K}{\omega_1}(w+\overline{b}+2a)\ +iK'\right]}{\Theta\left[\frac{K}{\omega_1}(w+\overline{b}+2a)\ +iK'\right]} +$$

$$\left.\frac{X+iY}{2\pi}h'_1(w) + \frac{X'+iY'}{2\pi}h'_2(w) + \frac{X-iY}{2\pi}h'_3(w) + \frac{X'-iY'}{2\pi}h'_4(w)\right\} \tag{B22}$$

where $h'_k(w) = \frac{d}{dw}h_k(w)$; $\quad k = 1, \ldots, 4.$

From reference 4 (p. 480),

$$\frac{H'(w)}{H(w)} = \frac{i\pi}{2K} + \frac{\Theta'(w+iK')}{\Theta(w+iK')}$$

so that on comparing equations (B22) and (13)

$$A_1 = \frac{1}{2\pi}\Big[h'_1(w)(X+iY) + h'_2(w)(X'+iY') +$$

$$h'_3(w)(X-iY) + h'_4(w)(X'-iY')\Big] \tag{B23}$$

By using equation (B23), equation (B21) can be rewritten as

$$W(w) = W(e^w) = \frac{X+iY}{2\pi} \log_e \Theta\left[\frac{K}{\omega_1}(w - b) + iK'\right] +$$

$$\frac{X-iY}{2\pi} \log_e \Theta\left[\frac{K}{\omega_1}(w+\bar{b}+2a)+iK'\right] + \frac{X'+iY'}{2\pi} \log_e \Theta\left[\frac{K}{\omega_1}(w-b')+iK'\right] +$$

$$\frac{X'-iY'}{2\pi} \log_e \Theta\left[\frac{K}{\omega_1}(w+\bar{b}'+2a) + iK'\right] + A_1 w \tag{14}$$

APPENDIX C

FORMULATION OF FUNCTION $Q(z)$

In order to make certain that the conditions on $g(z)$ (equations (32) and (33)) are satisfied, a function $Q(z)$ is introduced as follows:

Let

$$g(z) = Q(z)\ h(z) \tag{C1}$$

where $h(z)$ is analytic in the annular region and $Q(z)$ has zeros at $Re^{i\gamma}$ and 1. Furthermore, in order that the real and the imaginary parts of $h(z)$ may be separated later, $Q(z)$ must be real on the boundary of the annular region.

Let

$$z = e^{w}$$

and

$$Q(w) \equiv Q[z(w)] \tag{C2}$$

Consider the product

$$\left\{H\left(w\ \frac{K}{\omega_1}\right)H\left[\frac{K}{\omega_1}(w - r - i\gamma)\right]\ H\left[\frac{K}{\omega_1}(w + 2a)\right]\ \ H\left[\frac{K}{\omega_1}(w + 2a + r - i\gamma)\right]\right\}$$

where (reference 8, ch. XXI)

$$H(w) = \vartheta_1\left(w\vartheta_3^{-2}\mid\tau\right) \tag{C3}$$

and

$$\Theta(w) = \vartheta_4\left(w\vartheta_3^{-2}\mid\tau\right) \tag{C4}$$

From reference 4 (p. 487),

$$\vartheta_1(y + z)\vartheta_1(y - z)\vartheta_4^{\ 2} = \vartheta_1^{\ 2}(y)\vartheta_4^{\ 2}(z) - \vartheta_4^{\ 2}(y)\vartheta_1^{\ 2}(z) \tag{C5}$$

so that if

$$y = (w + a) \frac{\pi}{2\omega_1}$$

$$z = \frac{-a\pi}{2\omega_1}$$

Then

$$\vartheta_1 \left(w \frac{\pi}{2\omega_1}\right) \vartheta_1 \left[(w + 2a) \frac{\pi}{2\omega_1}\right] \vartheta_4^{\,2} = \vartheta_1^{\,2} \left[(w + a) \frac{\pi}{2\omega_1}\right] \vartheta_4^{\,2}\left(\frac{-a\pi}{2\omega_1}\right) -$$

$$\vartheta_1^{\,2}\left(\frac{-a\pi}{2\omega_1}\right) \vartheta_4^{\,2} \left[(w + a) \frac{\pi}{2\omega_1}\right] \tag{C6}$$

or

$$H\left[\frac{K}{\omega_1} (w)\right] H\left[\frac{K}{\omega_1} (w + 2a)\right] \Theta^2(0) = H^2\left[\frac{K}{\omega_1} (w + a)\right] \Theta^2\left[\frac{-Ka}{\omega_1}\right] -$$

$$H^2\left[\frac{-Ka}{\omega_1}\right] \Theta^2\left[\frac{K}{\omega_1} (w + a)\right] \tag{C7}$$

Similarly,

$$H\left[\frac{K}{\omega_1} (w - r - i\gamma)\right] H\left[\frac{K}{\omega_1} (w + 2a + r - i\gamma)\right] \Theta^2 (0)$$

$$= H^2\left[\frac{K}{\omega_1} (w + a - 2i\gamma)\right] \Theta^2\left[\frac{-K}{\omega_1} (a + r)\right] -$$

$$H^2\left[\frac{-K}{\omega_1} (a + r)\right] \Theta^2\left[\frac{K}{\omega_1} (w + a - 2i\gamma)\right] \tag{C8}$$

so that for $z = Ae^{i\theta}$ or $w = -a + i\theta$ equations (C7) and (C8) are both real.

Therefore, if

$$Q(w) = \left\{ H\left[\frac{K}{\omega_1}\,(w)\right] H\left[\frac{K}{\omega_1}\,(w - r - i\gamma)\right] H\left[\frac{K}{\omega_1}\,(w + 2a)\right] H\left[\frac{K}{\omega_1}\,(w - r + 2a - i\gamma)\right] \right\} \tag{C9}$$

$Q\left[w(z)\right]$ has zero value at $z = 1$ and $z = Re^{i\gamma}$ and takes on real values on the circle $|z| = A$.

For computation along $|z| = C$, write $Q(w)$ in the form

$$Q(w) = \left\{ H\left[\frac{K}{\omega_1}\,(w)\right] H\left[\frac{K}{\omega_1}\,(w - r - i\gamma)\right] H\left[\frac{K}{\omega_1}\,(w - 2c)\right] H\left[\frac{K}{\omega_1}\,(w - 2c + r - i\gamma)\right] \right\} \tag{C10}$$

This equation can be expressed in a form similar to that of equation (C9).

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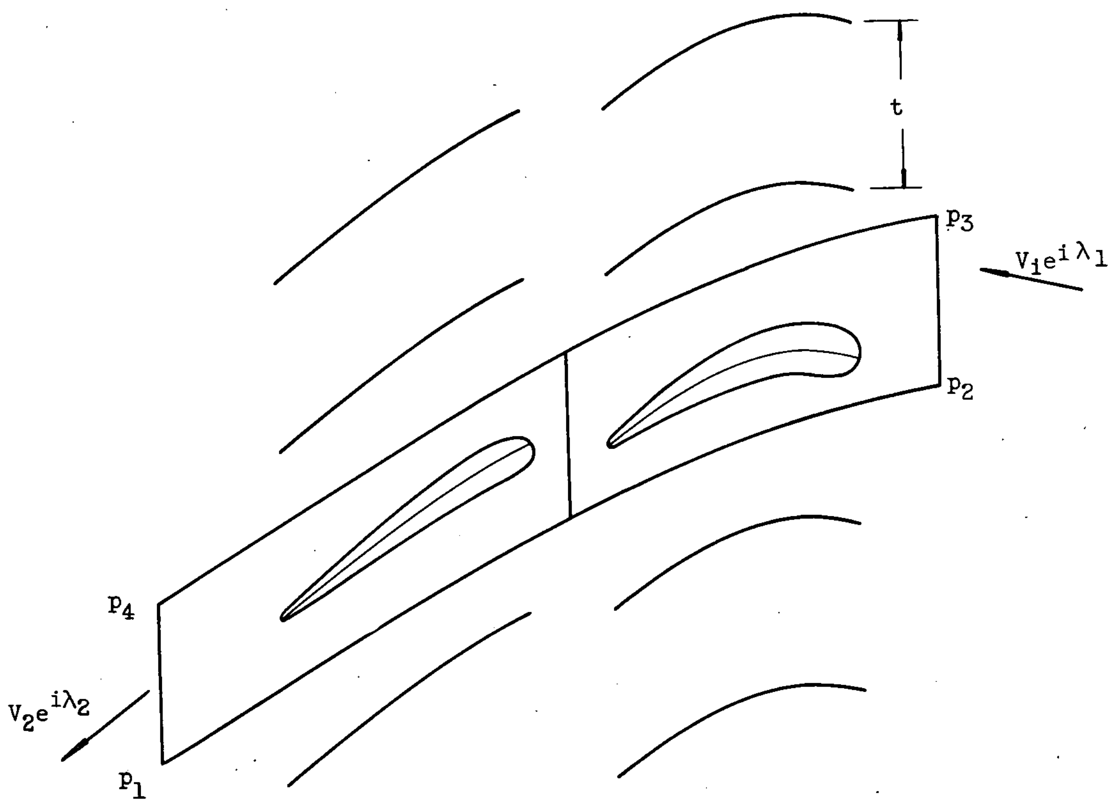
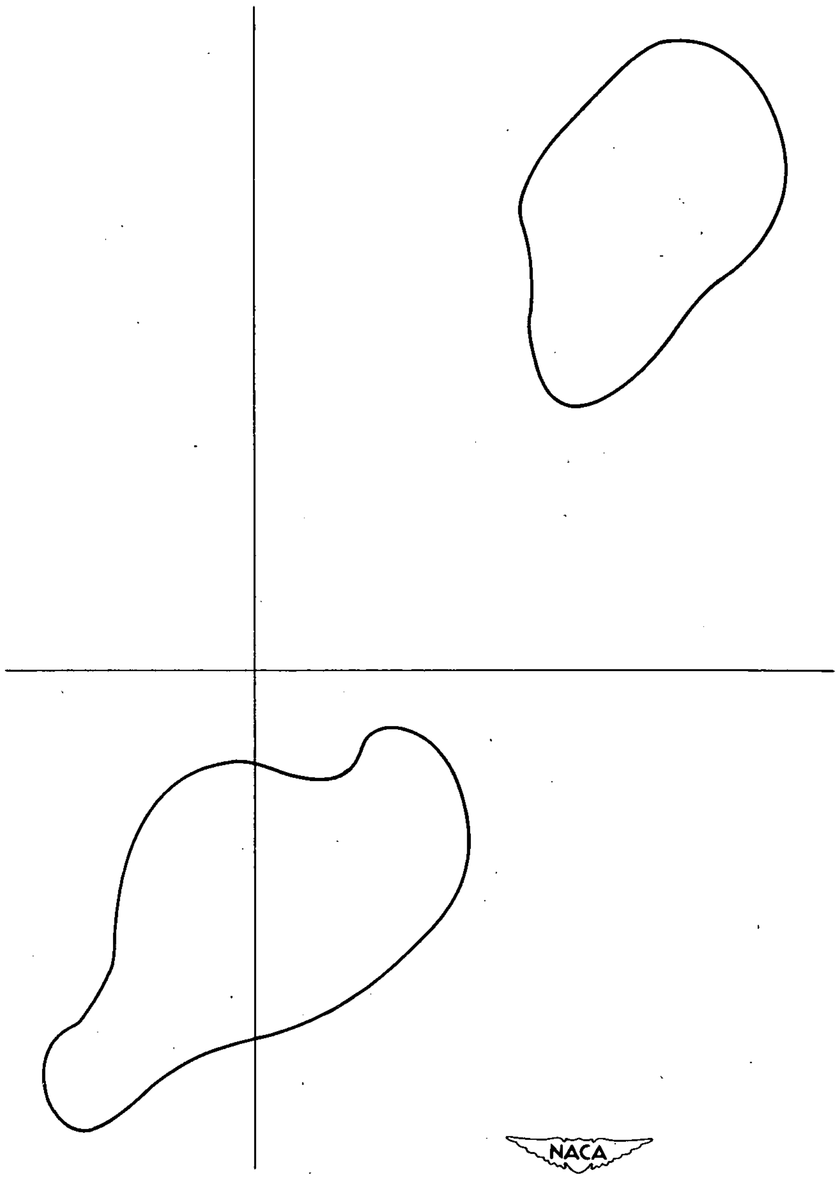


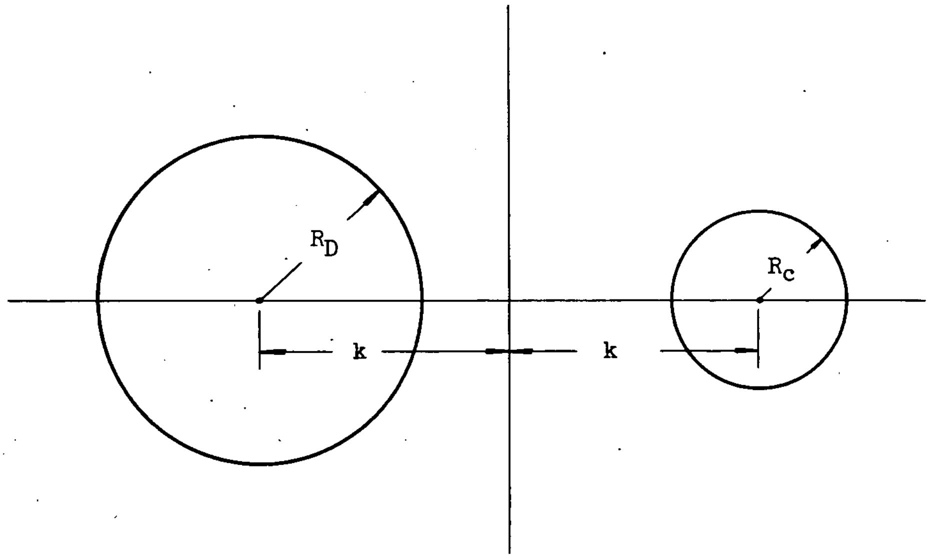
(a) σ-plane.

Figure 1. - Stages of conformal mapping.

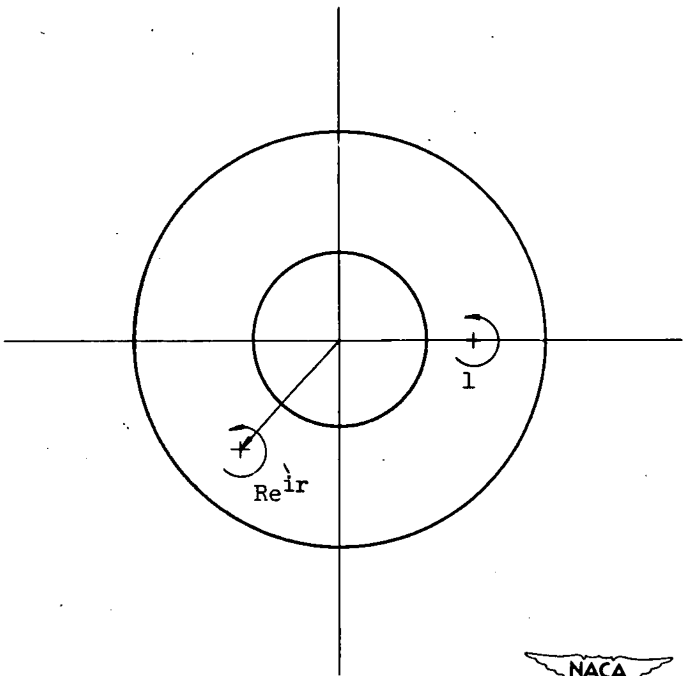


(b) ξ-plane.

Figure 1. - Continued. Stages of conformal mapping.

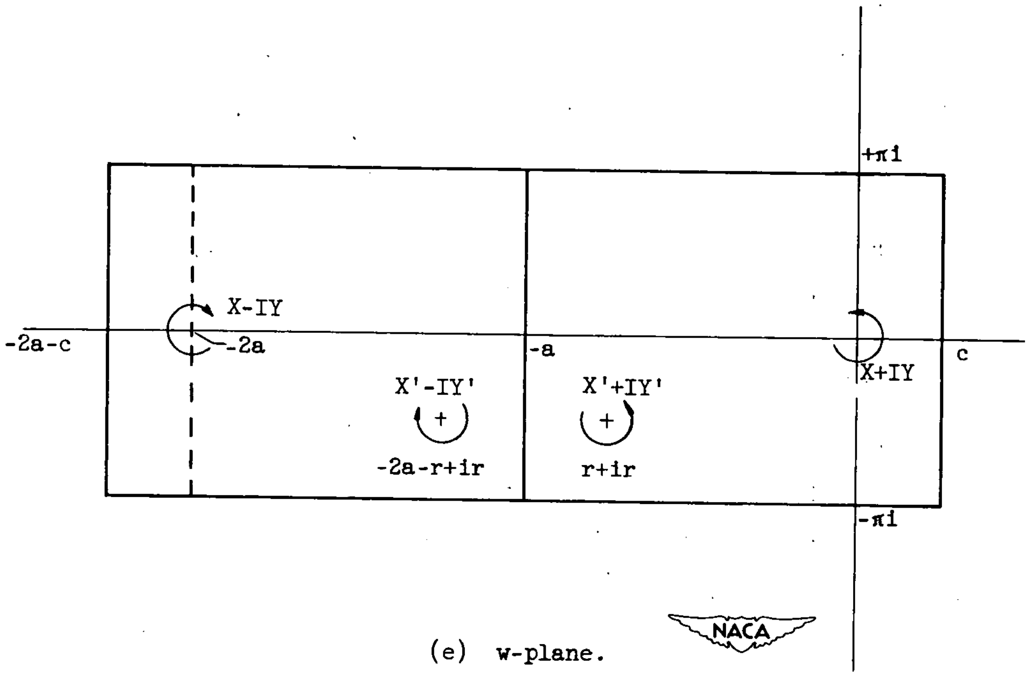


(c) ζ-plane.



(d) z-plane.

Figure 1. - Continued. Stages of conformal mapping.



(e) w-plane.

Figure 1. - Concluded. Stages of conformal mapping.

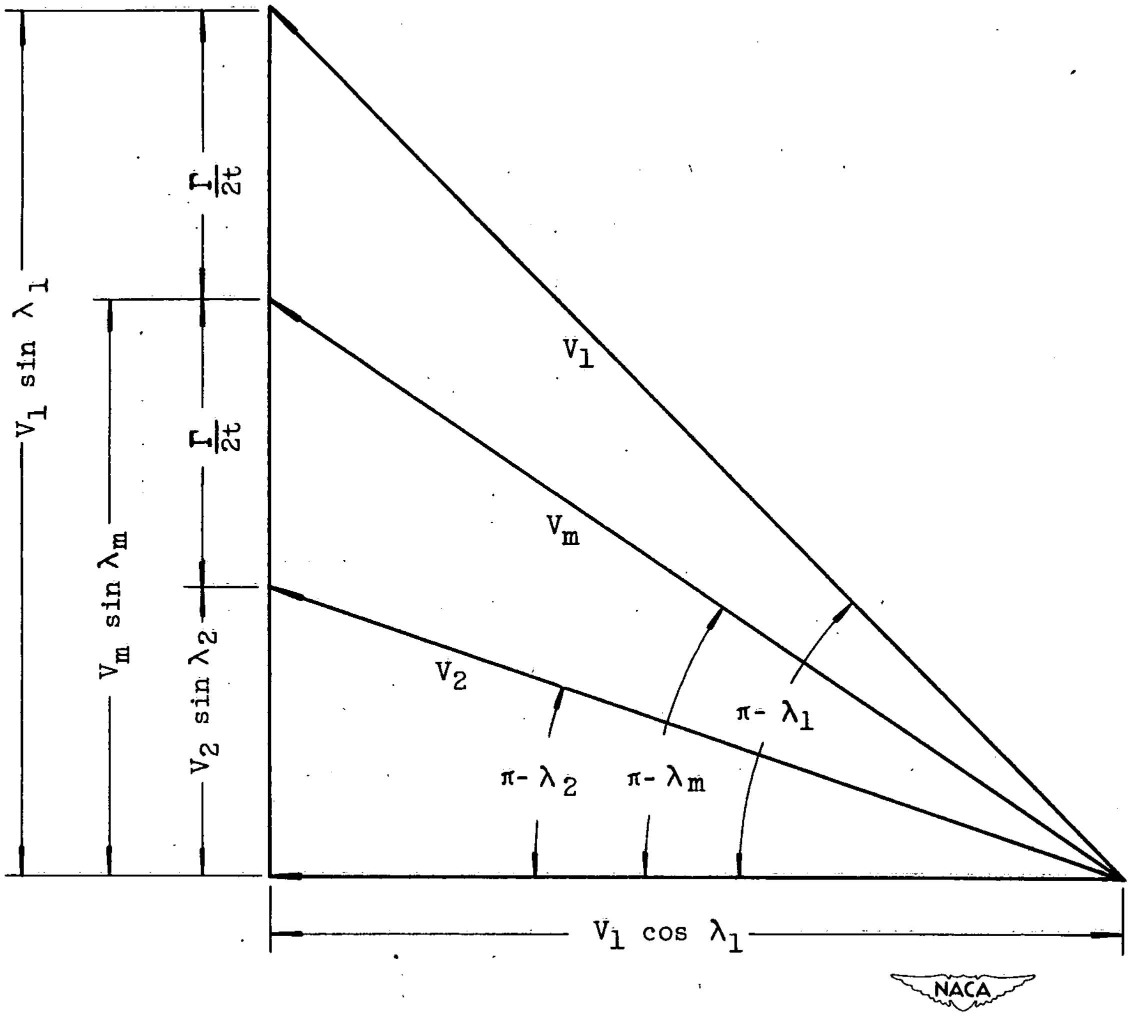
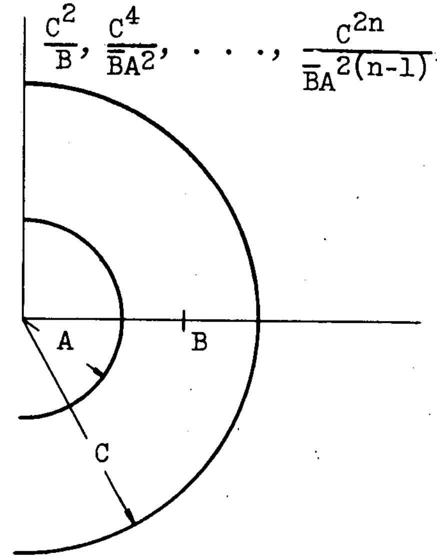


Figure 2. - Vector velocity diagram.

Singularity positions -- First reflection in outer circle

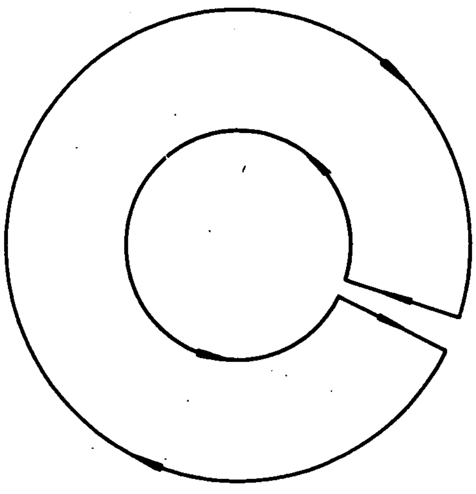
$$\ldots, \frac{BA^{2n}}{C^{2n}}, \ldots, \frac{BA^4}{C^4}, \frac{BA^2}{C_2} \quad \frac{C^2}{B}, \frac{C^4}{BA^2}, \ldots, \frac{C^{2n}}{BA^{2(n-1)}}, \ldots$$



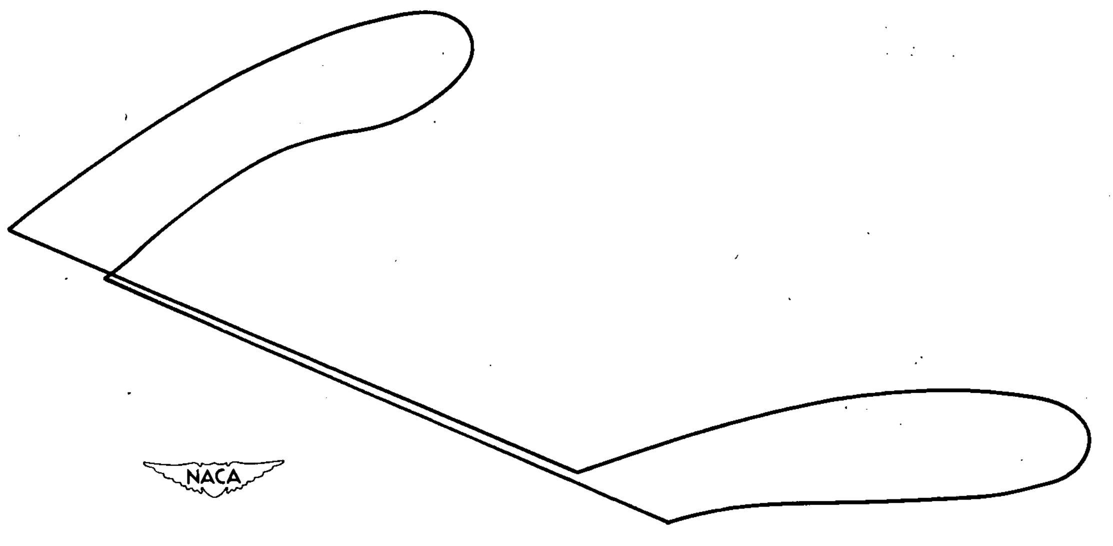
$$\ldots, \frac{A^{2n}}{BC^{2(n-1)}}, \ldots, \frac{A^4}{BC^2}, \frac{A^2}{B} \quad \frac{BC^2}{A^2}, \frac{BC^4}{A^4}, \ldots, \frac{BC^{2n}}{A^{2n}}, \ldots$$

Singularity positions -- First reflection in inner circle

Figure 3. - Positions of reflected singularities.



(a) Path of integration.



(b) Profile corresponding to figure 4(a).

Figure 4. - Illustration of possible lack of closure.